AustMS Talk

6/11/22

A knot-theoretic approach to comparing the Grothendieck-Teichmüller and Kashiwara-Vergne groups

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joint with
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Outline

- Two parallel stories:
  - algebraic (Drinfel'd, Kashiwara, Vergne, Alekseev, Enriquez, Torossian, ...)
  - knot-theoretic (Bar-Natan, Dancso)
- Completing part of the construction on the knot-theoretic side
- Showing equivalence of the new construction on the algebraic side

The algebra story

[Drinfel'd, '90] [Kashiwara–Vergne, '78] [Alekseev–Meinrenken, '05] [Alekseev–Torossian, '12] [Alekseev–Enriquez–Torossian, '10]

$$\begin{tikzcd}
KV \arrow[r, phantom, "\circlearrowright"] & \{\text{KV-Solutions}\} & KRV \arrow[l, phantom, "\circlearrowleft"] \\
GT \arrow[u] \arrow[r, phantom, "\circlearrowright"] & \{\text{Drinfel'd associators}\} \arrow[u] & GRT \arrow[u] \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}$$

The topology story

Let A be an augmented algebra and let $I = \ker(\varepsilon : A \to \mathbb{K})$.

Def. $gr(A) := \bigoplus_n I^n / I^{n+1}$

Def. An expansion is a map $Z: A \to gr(A)$ such that $gr(Z) = id$.

This is equivalent to saying that $Z|_{I^m}$ is the quotient map $I^m \to I^m / I^{m+1}$.

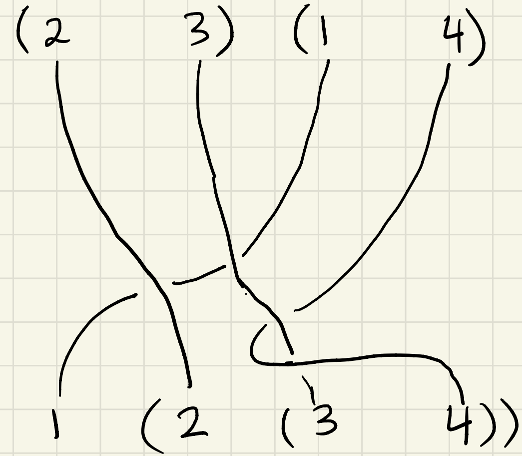
Def. A homomorphic expansion of A is an expansion which commutes with all the operations of A.

The topology story

[Bar-Natan, '95]

Parenthesised braids (PaB)

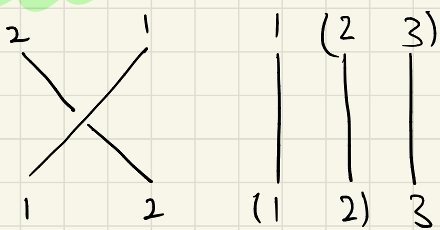
Operad consisting of groupoids PaB(n) with objects PaP(n) and morphisms PB_n.



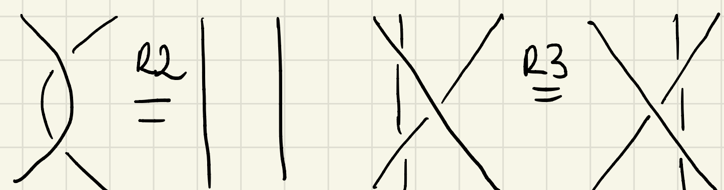
Operations

- stacking
- operadic composition ('cabling')

Generators



Relations

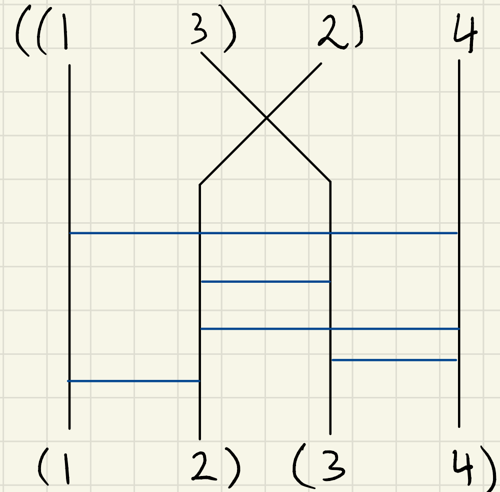


The topology story

[Bar-Natan, '95]

Parenthesised chord diagrams (PaCD)

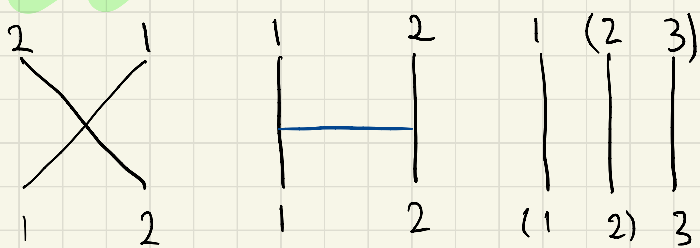
Operad consisting of groupoids PaCD(n) with objects PaP(n) and morphisms CD(n).



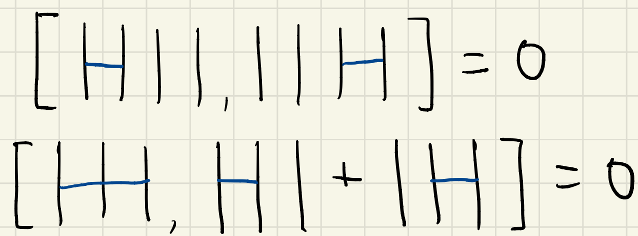
Operations

- stacking
- operadic composition ('cabling')

Generators



Relations



The topology story

[Bar-Natan, '95]

homomorphic expansions

$$\mathrm{Aut}(\mathrm{PaB}) \circlearrowright \mathrm{PaB} \; \{ \longrightarrow \} \; \mathrm{PaCD} \circlearrowleft \mathrm{Aut}(\mathrm{PaCD})$$

parenthesised braids

parenthesised chord diagrams

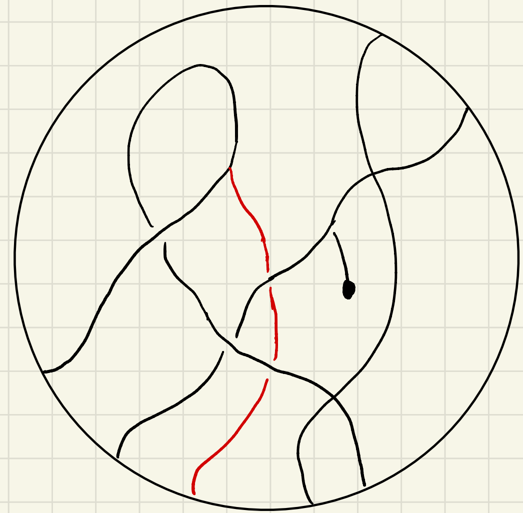
The topology story

[Bar-Natan–Dancso, '12]

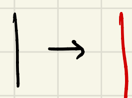
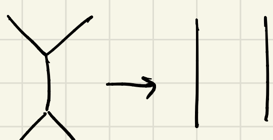
Welded foams (wF)

Circuit algebra of tangled surfaces and strings in $\mathbb{R}^4$.

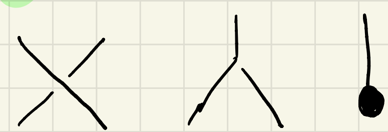
Represented by Reidemeister-like theory with surfaces $\uparrow$ and strings $\uparrow$.



Operations

- CA composition
- puncture 
- unzip 

Generators



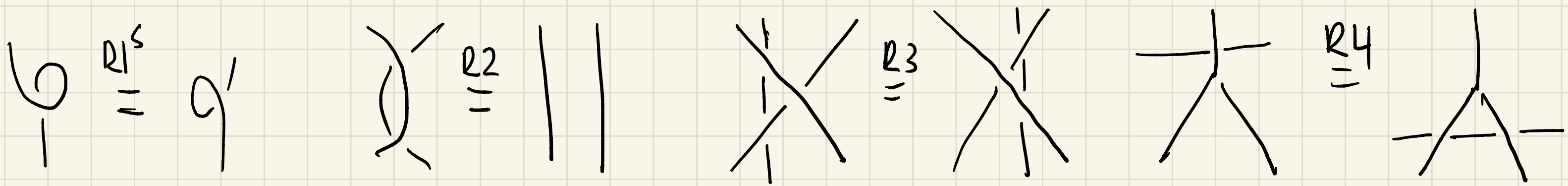
(+ allowable punctured variants)

The topology story

[Bar-Natan–Dancso, '12]

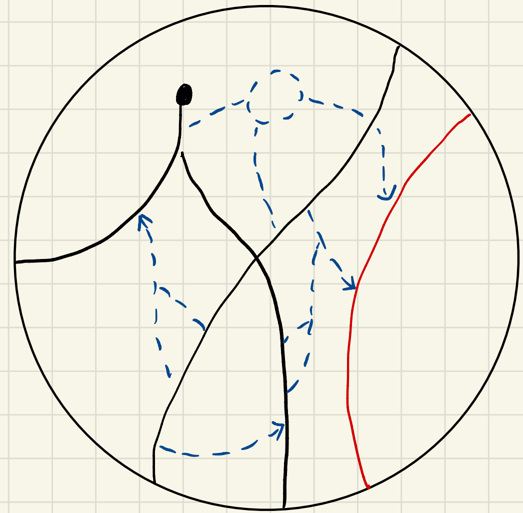
Welded foams (wF)

Relations



The topology story

[Bar-Natan–Dancso, '12]



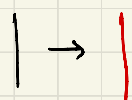
Welded arrow diagrams ($\mathcal{A}$)

Circuit algebra of Jacobi diagrams on w-foam skeleta.

Jacobi diagrams are uni-trivalent graphs with all univalent vertices ending on skeleton.

Operations

- CA composition
- puncture



- unzip

Generators

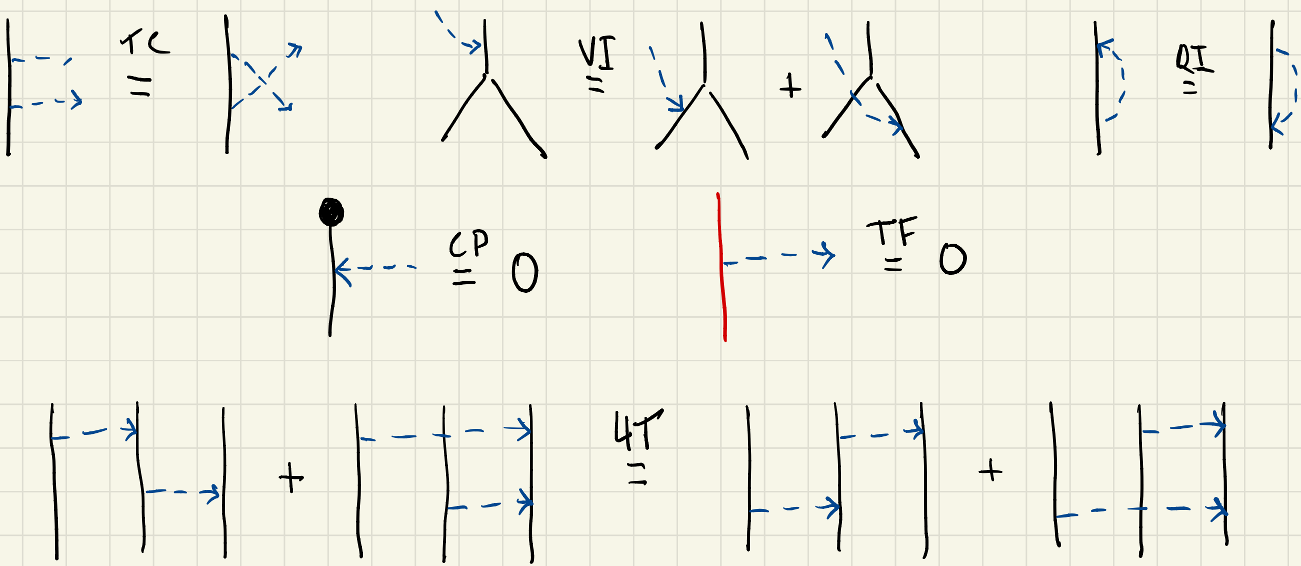
(+ allowable punctured variants)

The topology story

[Bar-Natan–Dancso, '12]

Welded arrow diagrams ($\mathcal{A}$)

Relations



The topology story

[Bar-Natan–Dancso, '12]

homomorphic expansions

$$\mathrm{Aut}(wF) \circlearrowright wF \;\; \{ \longrightarrow \} \;\; \mathcal{A} \circlearrowleft \mathrm{Aut}(\mathcal{A})$$

welded foams

arrow diagrams

State of play

$$\begin{tikzcd}
KV \arrow[r, phantom, "\circlearrowright"] & \{KV\text{-Solutions}\} \arrow[r, phantom, "\circlearrowleft"] & KRV \\
GT \arrow[u] \arrow[r, phantom, "\circlearrowright"] & \{\text{Drinfel'd associators}\} \arrow[u] \arrow[r, phantom, "\circlearrowleft"] & GRT \arrow[u]
\end{tikzcd}$$

$$\mathrm{Aut}(wF) \circlearrowright wF \ \{ \longrightarrow \} \ \mathcal{A} \circlearrowleft \mathrm{Aut}(\mathcal{A})$$

$$\mathrm{Aut}(PaB) \circlearrowright PaB \ \{ \longrightarrow \} \ PaCD \circlearrowleft \mathrm{Aut}(PaCD)$$

The algebra map

GRT and KRV are both subgroups of the group $TAut_2$.

Def. $TAut_2$ consists of automorphisms g of $U(lie_2)$ such that for each generator x_i,
$g(x_i) = g_i^{-1} x g_i$ for g_i in $\exp(lie_2)$.

An element of TAut can be represented by $(g_1(x,y), g_2(x,y))$ with g_i in $\exp(lie_2)$. Every element of GRT can be written as $(0, f(x,y))$ for some f in $\exp(lie_2)$.

Def. [AT]

$$\begin{aligned} \rho : \mathrm{GRT} &\longrightarrow \mathrm{KRV} \\ (0, f(x,y)) &\longmapsto (f(-x-y, x), f(-x-y, y)) \end{aligned}$$

Thm. [AT, '12] The above map is the unique injective group homomorphism that respects the bi-torsor structure on Drinfel'd associators and Kashiwara–Vergne solutions.

The knot-theoretic map

Aut(PaCD)

Expansion preserving automorphisms given by their value on the associator.

For F in Aut(PaCD), the value of F on the associator Φ is given by an element of GRT $(0, f(x,y))$ by taking

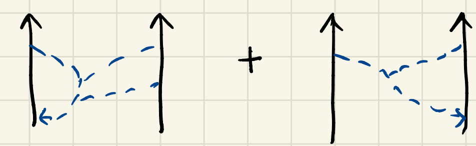
$$F(\Phi) = f(t^{12}, t^{23})$$

where t^{ij} is the chord from strand i to strand j.

Aut($\mathcal{A}$)

Expansion preserving automorphisms given by their value on the vertex ⅄ .

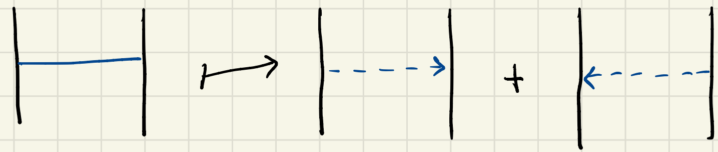
For G in Aut($\mathcal{A}$), the value of G on the vertex can be expressed as an element of $U(\mathrm{lie}_2^2)$ by reading off Lie words from trees, considering vertices as brackets.



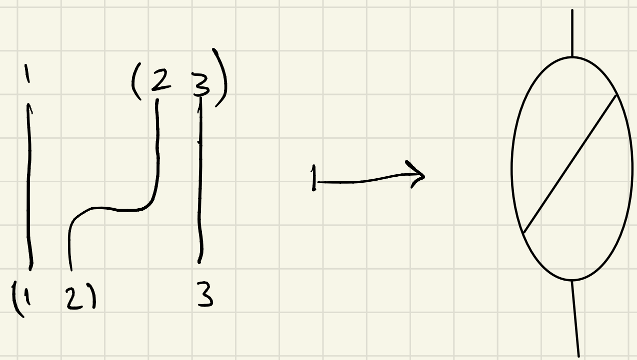
$$\big(\, [[x,y],y] \;,\; [x,y] \,\big)$$

The knot-theoretic map

There is a map $\alpha : \mathrm{PaCD} \longrightarrow \mathcal{A}$ which is defined by first



and then replacing the parenthesisation by binary trees representing the same data, as seen in the following example:



The knot-theoretic map

To construct a map $\tilde{\rho} : \mathrm{Aut}(\mathrm{PaCD}) \longrightarrow \mathrm{Aut}(\mathcal{A})$, we want to construct for every F in $\mathrm{Aut}(\mathrm{PaCD})$ a unique G such that the following commutes:

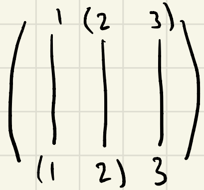
$$\begin{tikzcd}
\mathrm{PaCD} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{A} \arrow[d, dashed, "G"] \\
\mathrm{PaCD} \arrow[r, "\alpha"'] & \mathcal{A}
\end{tikzcd}$$

We construct the map G by exploiting two facts:

- if it exists, it satisfies the above diagram
- if it exists, it commutes with the operations of $\mathcal{A}$ (since it must preserve expansions)

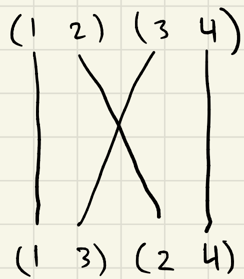
The knot-theoretic map

F in Aut(PaCD) is determined by its value

F 

$= f(t^{12}, t^{23})$.

There is a special chord diagram β

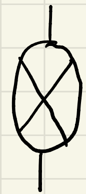
$\beta :=$ 

$= \left(\Phi^{(13)24}\right)^{-1} \Phi^{132} \left(\Phi^{123}\right)^{-1} \Phi^{(12)34}$

which has $F(\beta) = \left(f^{(13)24}\right)^{-1} f^{132} \left(f^{123}\right)^{-1} f^{(12)34}$.

The knot-theoretic map

Then,

$$\alpha(\beta) =$$  $$:= \beta^{cl}$$

And, we know that we must have

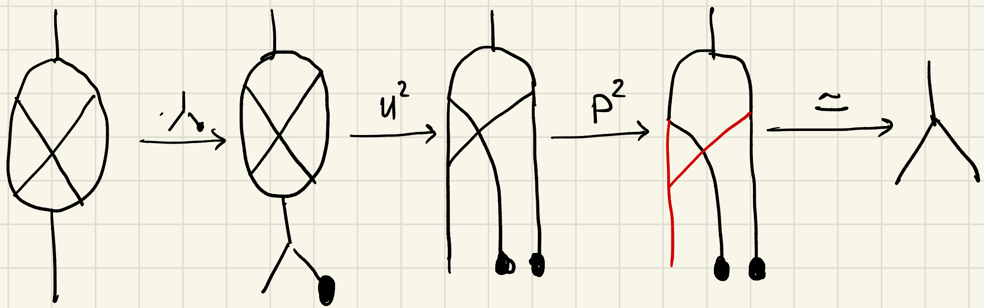
$$G(\beta^{cl}) = \alpha(F(\beta)).$$

We can then use $G(\beta^{cl})$ to determine what $G(\lambda)$ is, and therefore determine all of G.
But how!?

The knot-theoretic map

Short answer: we don't have time!

Long answer: G is required to commute with all the following series of operations, so we can extract the vertex value through applying these operations to the value $G(\beta^{c_1})$



Define the map $\tilde{\rho} : \mathrm{Aut}(\mathrm{PaCD}) \longrightarrow \mathrm{Aut}(\mathcal{A})$ by setting $\tilde{\rho}(F) = G$.

The knot-theoretic map

Prop. $\tilde{\rho}$ is an injective group homomorphism.

Thm. $\tilde{\rho}$ is equivalent to the AT map ρ.

$$\begin{tikzcd}
GRT \arrow[r, "\rho"] \arrow[d, "\cong"'] & KRV \arrow[d, "\cong"] \\
\mathrm{Aut}(PaCD) \arrow[r, "\tilde{\rho}"'] & \mathrm{Aut}(\mathcal{A})
\end{tikzcd}$$